EXPERIENCES SIMULATING NONLINEAR INTEGRABLE OPTICS*

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Abstract

With increasing interest in the nonlinear integrable optics, it is important that early experiences with simulating the lattices be shared to save time and point out potential difficulties in the simulations. We present here some details of simulating the nonlinear integrable lattices. We discuss correctly implementing and testing the elliptic element kicks, and the limits of the thin lens approximation. We also discuss generating a properly matched bunch in the transverse phase space, and how to analyze the resulting computational data from simulations.

INTRODUCTION

Nonlinear integrable optics [1] is a concept for mitigating collective instabilities in intense beams. It has generated considerable interest in the field, and multiple groups and institutions have begun contributing to the research. Because of the novel form of the potential, they require special considerations for beam matching, calculating the kick in a drift-kick symplectic tracking code, and analyzing the results of simulations.

At the moment, the nonlinear element is implemented and benchmarked in Synergia [2], LIFETRAC, PyORBIT [3], and MAD-X [4] by way of PTC [5]. Please contact the authors if you plan to implement the nonlinear element in your tracking code. As the collaboration grows and the number of implementations expands, it is important to keep track of updates and changes and compare computational results. By sharing post-processing scripts and source code, we hope to minimize the redundant effort required for new researchers to begin studying nonlinear integrable optics.

MATCHED DISTRIBUTIONS

The Hamiltonian described by eqns. (10) and (18) in the original paper by Danilov and Nagaitsev [1] is an invariant of the single-particle motion in a nonlinear integrable lattice. As is well-known, any distribution which is a pure function of the invariants is itself an invariant of the motion – it is a matched beam. Any computational study of a nonlinear integrable lattice must start with, at the minimum, a beam properly matched in the transverse direction including the elliptic potential. Early attempts at simple linear matching showed severe mismatch in the nonlinear lattice, leading to large excursions of single particle orbits.

The Hamiltonian in the normalized coördinates is given by

$$\mathcal{H} = \frac{\hat{p}_x^2}{2} + \frac{\hat{p}_y^2}{2} + \frac{\hat{x}^2}{2} + \frac{\hat{y}^2}{2} + t\mathcal{U}\left(\hat{x}, \hat{y}\right) \tag{1}$$

where $\mathcal{U}$ is the normalized elliptic potential given by eqn. (18) of [1]. Here, the overhat denotes the usual Courant-Snyder normalization of the momenta and coördinates. Thus, a function of $\mathcal{H}$ will be matched to a lattice with zero collective effects and zero longitudinal-transverse coupling (i.e. the zero chromaticity limit).

The generalization of the Kapchinskij-Vladimirskij distribution [6] is a delta function in the generalized emittance $f(\varepsilon) = \delta(\varepsilon - \varepsilon_0)$ for a beam where every particle has its Hamiltonian equal to a single emittance, $\mathcal{H}_i = \varepsilon_0$. To generate this delta function distribution, it is convenient to pick a magnitude of the transverse momentum, $\hat{p}_0$, at random, limited to the range $\hat{p}_0 \in [0, \sqrt{2\varepsilon_0})$. Then, solve for $\hat{x}$ and $\hat{y}$ in the nonlinear equation

$$\frac{1}{2}\left(\hat{x}^2 + \hat{y}^2\right) + t\mathcal{U}(\hat{x}, \hat{y}) = \varepsilon_0 - \frac{\hat{p}_0^2}{2} \tag{2}$$

One method of doing this is to pick an angle at random from $\theta \in [0, 2\pi)$ and solve for the radius using a numerical root finding algorithm. Another method, what we call the "lemming method", is to pick random values of $\hat{x}$ and $\hat{y}$ from inside some bounding box and walk the point in $\hat{x}$ at fixed $\hat{y}$ until it solves the origin. Either method should generate a uniform distribution in the $\hat{x}-\hat{y}$ plane, filling the isoenergetic contours in Fig. (1).

To generate a general distribution in $\mathcal{H}$, generate the distribution as a Riemann sum of delta functions on the desired distribution function. Thus, $f(\varepsilon)$ is a Riemann sum of K-V distributions. Specifically, consider the unit-normalized distribution $f(\varepsilon)$, such that

$$\int_0^\infty d\varepsilon\, f(\varepsilon) = 1 \tag{3}$$

and a total of N particles. Then on a finite interval from $\varepsilon' - \Delta\varepsilon/2$ to $\varepsilon' + \Delta\varepsilon/2$, there will be $Nf(\varepsilon')\Delta\varepsilon$ total particles. If there are $N_{\text{macro.}}$ macroparticles with weight w, then there will be $N_{\text{macro.}} w f(\varepsilon')$ macroparticles with $\mathcal{H} = \varepsilon'$. Thus, a way to generate the arbitrary distribution $f(\varepsilon)$, we can approximate this with a large number of KV distributions with the proper number of macroparticles in each sub-distribution.

To invert the generated coördinates to real coördinates, generate a uniformly random angle θ for the momentum and

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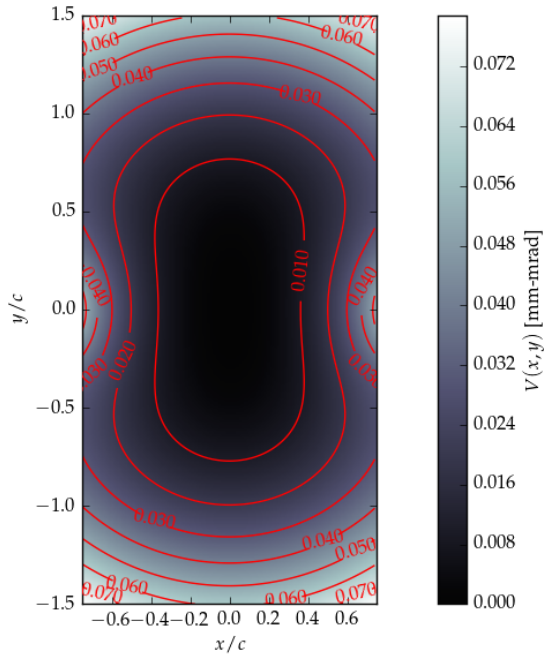

Figure 1: Isoenergetic contours for the elliptic potential. A K-V type distribution will uniformly fill the space inside a fixed energy contour. Here $t = -0.4$.

use the inversion of the Courant-Snyder parameterization:

$$\begin{aligned} x &= \hat{x}\sqrt{\beta_x} \quad & p_x &= \frac{\hat{p}_0 \cos\theta}{\sqrt{\beta_x}} - \alpha_x x \\ y &= \hat{y}\sqrt{\beta_y} \quad & p_y &= \frac{\hat{p}_0 \sin\theta}{\sqrt{\beta_y}} - \alpha_y y \end{aligned} \tag{4}$$

where here α and β are the Twiss parameters at the point of observation in the lattice.

COMPUTING THE KICK & INVARIANTS

Implementation and optimization of the kick and invariant calculations will be language-specific, and therein t and c are the normalized t and c, not those associated with the local magnet and Twiss parameters. They are the t that appears in eqn. (1) above, for example. A Python class which computes the kick and invariants from un-normalized coördinates may be found in the GitHub gist in [7]. The API for this is straightforward, and should be straightforward to adapt for any language or tracking code.

There will be a distinction between the ideal $\beta(s)$ in the elliptic element and the actual $\beta(s)$ for a lattice with misalignments, powering errors, and other physical effects. In all computational cases, one presumably scales the elliptic potential to the ideal $\beta(s)$, and so that is the one that should be used in all rescalings. We also note that, for gapped magnets, the numerical (and therefore, presumably, the physical) behavior is better if the magnet strength is scaled to the beta function in the middle of the magnet. A complete error analysis of this is beyond the scope of this proceeding.

The invariants are a critical tool for studying the dynamics of the beam under perturbations from the ideal lattice. This includes studying space charge, magnet errors, wake fields, and on. These invariants represent the ideal system, in the same way that the linear Courant-Snyder invariants represent the ideal system for linear strong focusing. Thus, useful diagnostics include calculating the distribution of H and I and how it evolves over time in the presence of collective effects or lattice errors.

CONVERGENCE & VERIFICATION

Tracking single particles across a nonlinear element is best done using second order drift-kick. In the actual construction of the elliptic magnet, the magnetic elements generating the elliptic potential will be separated by drifts. To accurately model this, it is necessary to carry out more than one drift-kick-drift sequence through the element. A single kick across the element, sandwiched between two drifts, will be indistinguishable to the tracking code from a magnet which is the full width of the magnet and gaps. In our experience, three kicks across a magnet is sufficient to get quantitative convergence and include the effects of gaps and magnet widths, while also minimizing the computational load. Because the kick features a substantial number of function evaluations, a poorly optimized kick could represent significant computational time.

To verify that the code has been implemented correctly, we suggest generating a particle with a fixed invariant using the method described above. Track the particle through an ideal ring which should represent double-focusing quadrupoles surrounding the elliptic element, as described in [1]. Both invariants should remain close to constant, as seen in Figs. (2) and (3).

Thus, a check on computing the invariants correctly is the shape of the potential isoenergetic contours in Fig. (1). This is particularly sensitive to sign errors and other bugs and thus the presence of any errors in the calculation of the Hamiltonian will be obvious. Once the correct computation of the Hamiltonian is established, carry out the aforementioned simulations. I and H should both remain close to constant, and H should be close to the initial value. If H has been confirmed as being computed correctly, and the value of H is varying by larger than about 1%, then there is an error in the calculation of the kick. If H is remaining well-behaved but I is varying by a large amount, then there is an error in the calculation of I.

Note the qualitative differences between the invariants in Fig. (2) and Fig. (3). While both show similar quantitative variation in the Hamiltonian and the invariant, the three-kick map shows a much more orderly structure in the variations of H and, especially, I over time. This may have important implications for longer term tracking with errors, although we have not studied these differences quantitatively.

An additional qualitative behavior is the Poincaré surfaces for a tracking. The single particle orbits tend to trace out an hourglass shape in the transverse $x - y$ and $p_x - p_y$ planes, as illustrated in Fig. (4).

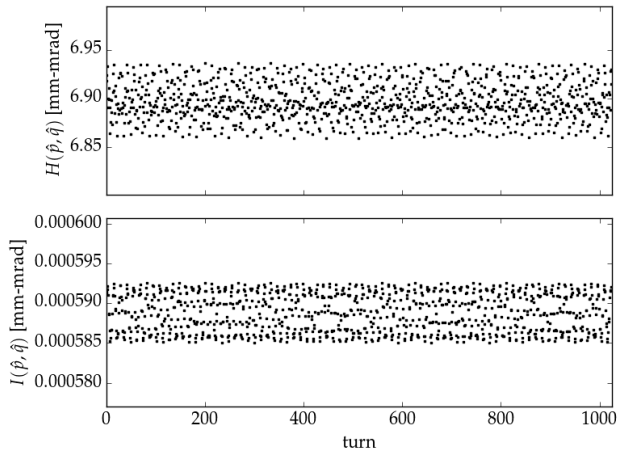

Figure 2: Invariants for an ideal gapped element with one kick per nonlinear magnet.

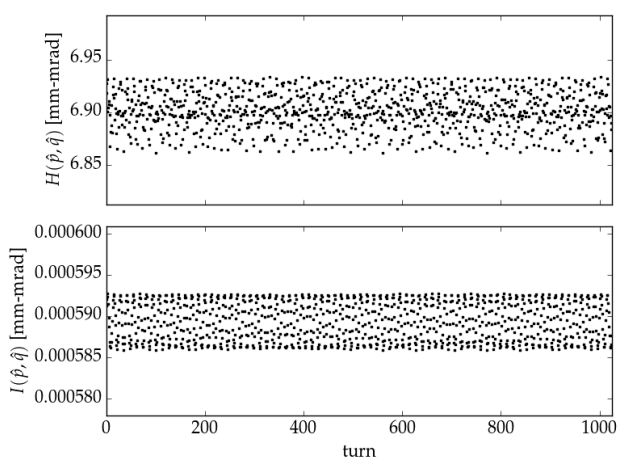

Figure 3: Invariants for an ideal gapped element with three kicks per nonlinear magnet.

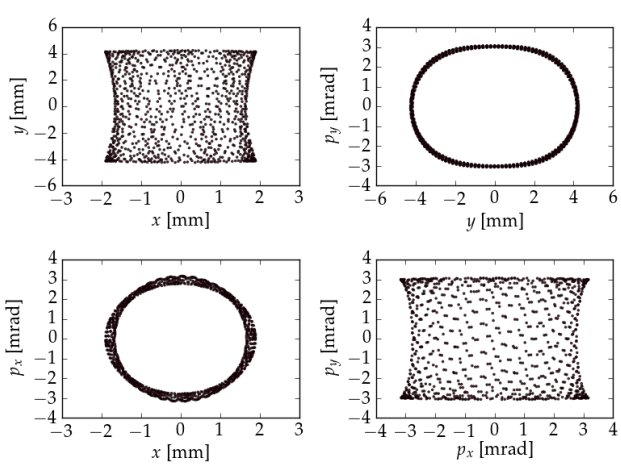

Figure 4: Poincaré sections for the transverse particle dynamics.

CONCLUSION

We have here provided documentation on GitHub of how to compute the kick and the invariants in a nonlinear integrable element, a discussion of heuristics for convergence for accurate simulations, and steps to verify the correct calculation of the invariant, Hamiltonian, and kick. We have also described conceptually how to generate a beam matched to the nonlinear elliptic lattice. We have provided guidelines from our experience with simulating these lattices, and suggested which diagnostics may prove to be meaningful for the nonlinear elements.

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